1 Introduction - Approaching geo problems

1. Organized and neat diagrams allow you to see both the big picture and the minute details.
2. Working both ways is a merit.
3. Staring at the problem isn’t going to solve it.
4. Making constructions is awesome. (e.g. parallel lines, circles, cyclic quads)

2 Brocard Points

We can show that inside any triangle ABC, there exists a unique point P such that

\[ \text{angle PAB = angle PBC = angle PCA} \]

This point is called the Brocard Point of triangle ABC

2.1 Proof of the Brocard Point

Let S be the center of the circumcircle of triangle ACP. Indeed if angle PAB and angle PCA were equivalent, then the circumcircle of triangle ACP is tangent to the line AB at A. Then, S lies on the perpendicular bisector of segments AC, and the line SA is perpendicular to AB.

Therefore, point P lies on the circle centered at S with radius SA (note that this circle is not tangent to line BC unless BA = BC). We can use the fact that angles PBC and PCA are equal to construct the circle passing through B and tangent to line AC at C. The Brocard point P must lie on both circles and be different from C. Such a point is unique. Therefore, the third equation (angle PAB = angle PBC) clearly holds.

2.2 Example Problem 1

[AIME 1999] Point P is located inside triangle ABC so that angles PAB, PBC, and PCA are all congruent. The sides of the triangles have lengths, AB = 13, BC = 14, and CA = 15, and the tangent of angle PAB is \(m/n\), where \(m\) and \(n\) are relatively prime positive integers. Find \(m + n\).
2.3 Example Problem 2

Let \( x, y, \) and \( z \) be positive real numbers satisfying the system of the equations

\[
3x^2 + 3xy + 3y^2 = 75 \\
y^2 + 3z^2 = 27 \\
z^2 + xz + x^2 = 16
\]

Evaluate \( xy + 2yz + 3xz \)

3 Exercises

1. Point O lies inside a irregular pentagon ABCDE. Let angle BAO = angle BCO, angle CBO = angle CDO, angle DCO = angle DEO, angle EDO = angle EAO. If angle AEO = 24, what are the possible values of the measure of angle ABO in degrees?

2. Prove that there is one and only one triangle whose side lengths are consecutive integers, and one of whose angles is twice as large as another.

3. Consider the five points A, B, C, D, E such that ABCD is a parallelogram and BCED is a cyclic quadrilateral. Let \( l \) be a line passing through A. Suppose that \( l \) intersects the interior of segment CD at \( F \) and intersects line BC at \( G \). Suppose that \( EF = EG = EC \). Prove that line \( l \) is the bisector of angle DAB.