Algebraic Manipulations & Inequalities

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Q1 Let \( f(x, y) \) be a function defined as

\[
f(x, y) = \frac{x + y}{\lfloor|x| + |y| + 1\rfloor}
\]

for all real numbers \( x \) and \( y \) where \( x, y > 0 \) and \( xy = 1 \). Find the all values that can be \( f(x, y) \).

**Answer:** \( \{\frac{1}{2}\} \cup \{\frac{5}{6}, \frac{7}{4}\} \)

**Proof**

Without loss of generality, let \( x \leq y \).

**Case 1** When \( x = y \), it is not difficult to show that \( f(x, y) \) can be simplified to \( f(x, y) = \frac{2x}{2|x| + 1} \). In this case, the possible value for \( f(x, y) \) is \( \frac{1}{2} \) since it can be seen that \( x = 1, y = 1 \) from the given relation that \( xy = 1 \).

**Case 2** When \( x \) is strictly less than \( y \), then we see that while \( x \) is less than 1, \( y \) is greater than 1. This suggests that \( |x| = 0 \), which simplifies the equation for \( f(x, y) \). The expression for \( f(x, y) \) becomes \( f(x, y) = \frac{x + 1}{|x| + 1} \). Using the fact that \( xy = 1 \), \( f(x, y) = \frac{x + 1}{x} \). Now, let \( x = p + q \) where \( p \) is a positive integer and \( q \) is the fractional part of the real number \( x \). Therefore, \( f(x, y) \) can be rewritten as \( f(x, y) = \frac{p + q + 1}{p + 1} \). It is not difficult to see that

\[
\frac{p + \frac{1}{2}}{p + 1} < \frac{p + q + 1}{p + 1} < \frac{p + q + \frac{1}{2}}{p + 1}
\]

It suffices to find values of \( n \) satisfying the above inequality. We work with the RHS of the inequality first. It can be rewritten as \( 1 + \frac{1}{(p+1)^2} \). Because we would like to maximize the value of the RHS where \( p \) is an integer, the greatest value of the RHS is when \( p = 1 \). Otherwise, the fraction, \( \frac{1}{(p+1)^2} \), will become smaller, as \( p \) increases. Therefore, the maximum value of \( f(x, y) = \frac{5}{4} \).

On the other hand, we must minimize the value of the LHS of the inequality. It can be rewritten as \( \frac{p + \frac{1}{2}}{p + 1} = \frac{(p + 1) - \frac{1}{2}}{p + 1} = 1 + \frac{-\frac{1}{2}}{p + 1} = 1 + \frac{-\frac{1}{2}}{p(p + 1)} \). It is not difficult to see that its minimum value is when \( p = 2 \). Therefore the minimum value of \( f(x, y) = \frac{3}{2} \). Therefore, we see that the values that satisfy can equal the function \( f(x, y) \) are \( \{\frac{1}{2}\} \cup \{\frac{5}{6}, \frac{7}{4}\} \).

Notes: It is important to make note of in this problem, the beginning approach to this problem. Specifically, notice how it is necessary to consider two different cases under the WLOG statement. Problems that ask you to prove an algebraic equation that may look intimidating at first can be simplified by organizing the variables into different cases.

Q2 Prove that \( (1 + \frac{1}{n})^n < \left(1 + \frac{1}{n+1}\right)^{n+1} \) for the positive integer \( n \).

**Proof**

We take the \((n + 1)^{th}\) root of both sides of the inequality. The LHS can thus be rewritten as \( \sqrt[n]{(1 + \frac{1}{n})^n} < \sqrt[n+1]{(1 + \frac{1}{n+1})^{n+1}} \). By the AM-GM inequality, the LHS is less than or equal to \( \frac{n(1 + \frac{1}{n}) + 1}{n+1} = 1 + \frac{1}{n+1} \). Therefore, \( (1 + \frac{1}{n})^n < \left(1 + \frac{1}{n+1}\right)^{n+1} \) as desired.

Notes: Observe the algebraic manipulation. When given a complicated, but short inequality like the question above, usually a clever manipulation is all that it takes to dumb down the problem. Foresight into these tricks can only be acquired through practice. So ... practice! These sample problem lectures can help you get there!